Exotic Integrals of the Arctangent function via Complex Contour Integration

Sid Gupta, Evan Pfeifer, Adrian Hito

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North Carolina School of Science and Mathematics
Durham, North Carolina, USA

Abstract

In this study, we examine the integral $\int_0^\infty \frac{\tan^{-1}(x)}{1+\alpha x+x^2}dx$ for real $\alpha > -2$, initially posed by Dr. Brian Bradie in the College Mathematics Journal. Traditional integration methods fail due to the complexity of combining the arctangent function with a rational expression, presenting a challenge as the solution requires a closed-form expression in terms of α. By adopting complex analysis, we discovered a novel representation and parameterization of the arctangent function through strategic substitution using the contour $1+ti$. Applying the Cauchy Residue Theorem, we linked integral evaluations over complex contours to the sums of residues at function poles, simplifying the integral's expression with the aid of the dilogarithm function. Our approach not only circumvents the limitations of directly applying the complex arctangent to yield a closed-form expression in terms of α, but also innovates on the idea of advanced complex analysis techniques in solving complex integrals involving arctangent.

Contents

1 Introduction

1261. *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.* Evaluate the following integral for real $\alpha > -2$:

$$\int_0^\infty \frac{\tan^{-1}(x)}{1+\alpha x + x^2}dx$$

Initially, we visualized the problem by graphing the integral's behavior, particularly noting the asymptote at $\alpha = -2$. This graphical analysis, supported by a curve fit, indicated a potential for a closed-form solution, as traditional methods such as integration by parts and Feynman's technique prove inadequate for such an arctangent-rational expression combination. Our exploration employs a novel method of complex contour integration, initially generating numerical values for each α to support our analytical approaches.

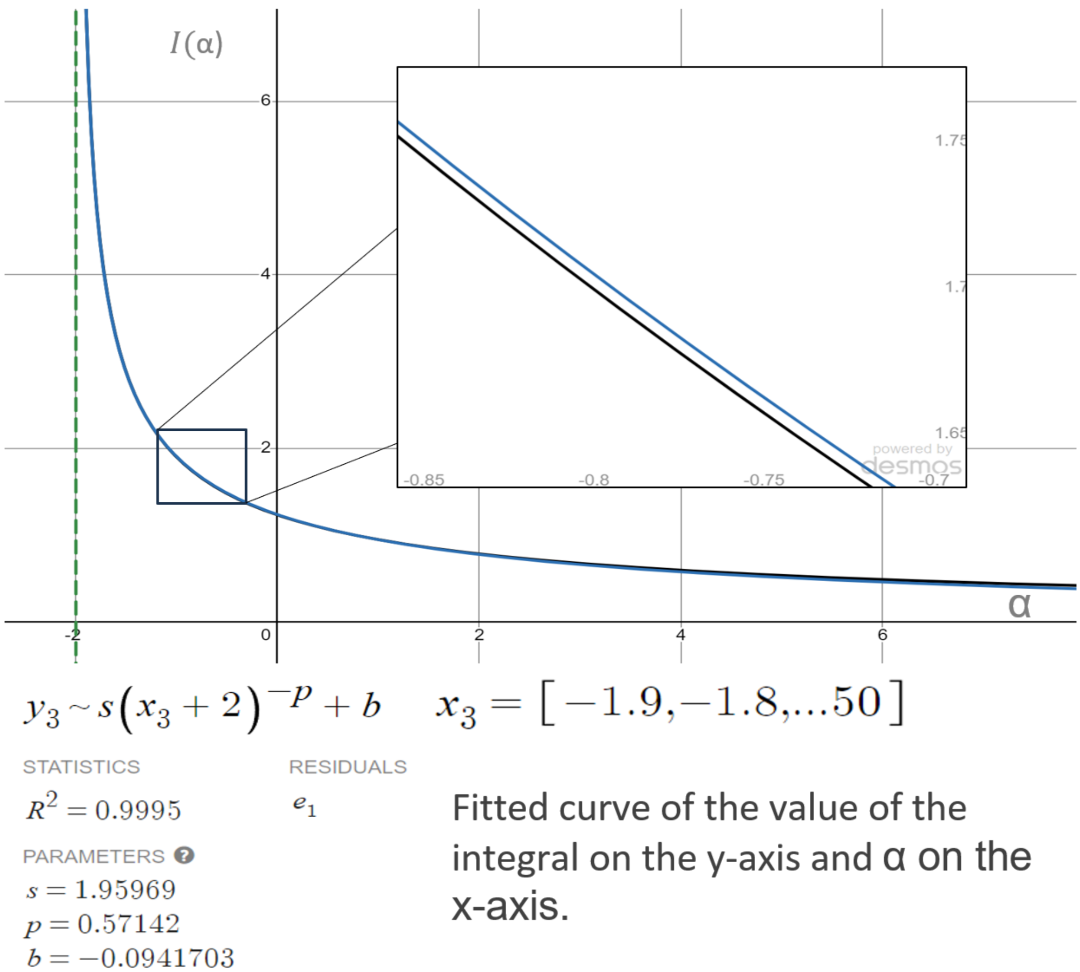


Figure 1: Fitted curve of $I(\alpha)$ on the y-axis and α on the x-axis, demonstrating the asymptotic behavior and suggesting the potential for closed-form solutions.

With the employed curve fit suggesting a closed-form solution, we further investigated specific cases of α:

$$\alpha = 0$$

$$\int_0^\infty \frac{\arctan x}{1+x^2}\,dx = \frac{(\arctan x)^2}{2}\Big|_0^\infty = \frac{\pi^2}{8}$$

$$\alpha = 2$$

$$\int_0^{\infty} \frac{\arctan x}{1+2x+x^2}\, dx = \frac{1}{4}\left(-\ln(x^2+1)+2\ln(x+1)+\frac{2(x-1)\arctan x}{x+1}\right)\Big|_0^{\infty} = \frac{\pi}{4}.$$

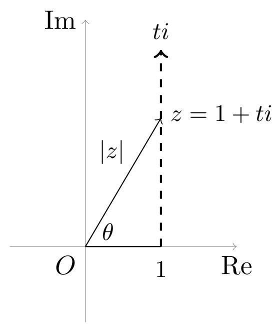
The initial drive to plugging in easier-to-work with alpha's was to establish a basic understanding of the interactions different α values have on the problem. From this, we can better understand strategies to approach the problem to begin generalizing for any α.

This integral is not amenable to traditional approaches such as integration by parts nor Feynman's method. In this problem, we explore a novel method of solving definite integrals involving arctangent in a rational expression, which connects arctangent to the complex logarithm using contour integration.

2 Methods

2.0 Defining a New Function

In the current form, the function $f(x) = \frac{\arctan(x)}{1+\alpha x+x^2}$ is difficult to work with. Our main motivation for exploring the problem in the context of the complex plane was that we thought we would be able to express $\arctan(x)$ in a different way in the complex plane, which would make the problem easier to approach. For instance, we noticed the complex inverse tangent function $\arctan(z) = \frac{1}{2i}\ln(\frac{i-z}{i+z})$. However, simply plugging it in and trying to manipulate the expression ws not effective.



$$f(z) = \frac{\ln(z)}{-z^2+(2-\alpha i)z+\alpha i}.$$

Plugging in $z = 1+ti \to t = -i(z-1)$, the denominator is equal. Our aim was that we could isolate the original to integral by integrating along the contour $C_1 = 1+ti$ for $t \in [0,\infty)$, and isolating the real part of the parameterized expression. We now need to find other contours to close the region such that we could use the Cauchy Residue Theorem to simplify it.

2.1 Cauchy Residue Theorem

The Cauchy Residue Theorem, the essential theorem behind contour integration, states that for a given $f(z)$ which is analytic in the region A except for a set of isolated poles, and C, a simple closed curve in A that doesn't go through any of the poles of f and is oriented counterclockwise,

$$\int_C f(z)dz = 2\pi i \cdot \sum(\text{poles of } f \text{ inside } C)$$

In this section, we will focus on setting up the right side of the equation, exploring different sets of contours and their respective integrals to see what lets us evaluate the LHS.

We will focus on the case where there are no poles in or on the curve (excluding endpoints, such as $0 + 0i$ in this case) formed by the set of contours, so that the RHS for the Residue Theorem simplifies to 0. To do this, we will start with the case where the imaginary part of z is negative, since the contours we are exploring are in the positive imaginary part of the complex plane.

We found an expression for poles by setting the denomiminator of $f(z)$, $-z^2 + (2 - \alpha i)z + \alpha i$, equal to 0 and solving for z with the quadratic formula to get an expression in terms of α. We split that solution into real and complex parts, and set the complex part negative to find the inequality representing α in that case. This inequality was

$$\sqrt[4]{\alpha^4 + 8\alpha^3 + 40\alpha^2 + 32\alpha + 16} \cdot \sin\left(\frac{1}{2}\arctan\left(\frac{-4\alpha}{\alpha^2 + 4\alpha + 4}\right)\right) < 0$$

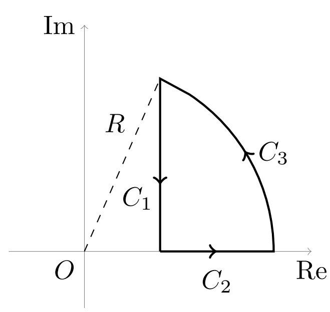
which simplified to,

$$\frac{4\alpha}{\alpha^2 + 4\alpha + 4} > 0$$

meaning that for this case, $\alpha > 0$, and it further suggests the asymptotic behavior around $\alpha = -2$.

2.2 First Contour Attempted

Let's explore this section of a circle, centered at 0 with radius r:



where $R \to \infty$,
$C_1 = 1 + ti$ for $t \in [0, \sqrt{R^2 - 1}]$,
$C_2 = t$ for $t \in [1, R]$,
$C_3 = Re^{ti}$ for $t \in [0, \arccos \frac{1}{R}]$,

The contour C_1 integrates along the line $C_1 = 1 + ti$, and after parameterised,

$$\int_{C_1} f(z)\,dz = \int_0^\infty f(1 + ti) \cdot \frac{d}{dt}[1 + ti]\,dt$$
$$= \int_0^\infty \frac{\arctan(t) - \ln\sqrt{1 + t^2}i}{1 + \alpha t + t^2}\,dt$$
$$\int_{C_1} f(z)\,dz = \int_0^\infty \frac{\arctan(t) - \ln\sqrt{1 + t^2}i}{1 + \alpha t + t^2}\,dt$$

Then, isolating the real part,

$$\mathrm{Re}\left(\int_0^\infty \frac{\arctan(t) - \ln\sqrt{1 + t^2}i}{1 + \alpha t + t^2}\,dt\right) = \int_0^\infty \frac{\arctan(t)}{1 + \alpha t + t^2}\,dt$$

It represents the integral from the original problem. C_2 integrates along the real axis. To start with this, we used partial fractions to separate $f(z)$, so that for
$c_1 = 1 - \frac{\alpha i}{2} + \frac{1}{2}\sqrt{4-\alpha^2}$ and $c_2 = 1 - \frac{\alpha i}{2} - \frac{1}{2}\sqrt{4-\alpha^2}$,

$$f(z) = \frac{1}{\sqrt{4-\alpha^2}}\left(\frac{\ln z}{z-c_1} - \frac{\ln z}{z-c_2}\right)$$

This function cannot be integrated in elementary functions, so it requires the introduction of a special function called the dilogarithm. Also known as Spence's function, The dilogarithm $\mathrm{Li}_2(z)$ is a special case of the polylogarithm $\mathrm{Li}_n(z)$ for $n = 2$. Polylogarithms have deep connections to exploration of the Riemman Zeta Function, Feynman Integrals in quantum field theory, combinatorics, and are generally extremely useful special functions. It can be expressed as

$$\mathrm{Li}_2(z) = \int_z^0 \frac{\ln(1-u)}{u} du$$

or

$$\mathrm{Li}_2(1-z) = \int_1^z \frac{\ln(u)}{1-u} du.$$

Integrating $f(z)$ along $C_2 = t$ for $t \in [1, 1+r]$ is simple since it is along the real axis, so it does not need to be parameterized.

$$\int_{C_2} f(z)dz = \int_1^\infty f(t) \cdot \frac{d}{dt}[t]\, dt$$

$$= \int_1^\infty \frac{\ln(t)}{-t^2 + (2-\alpha i)t + \alpha i}\, dt$$

$$\int_{C_2} f(z)dz = \int_1^\infty \frac{\ln(t)}{-t^2 + (2-\alpha i)t + \alpha i}\, dt$$

$$\int_1^r \frac{1}{\sqrt{4-\alpha^2}}$$

$$\left(\frac{\ln z}{z-c_1} - \frac{\ln z}{z-c_2}\right) dz$$

$$= \frac{1}{\sqrt{4-\alpha^2}}\left(\mathrm{Li}_2\left(\frac{r}{c_1}\right) - \mathrm{Li}_2\left(\frac{r}{c_2}\right) + \ln(r)\left(\ln\left(1-\frac{r}{c_1}\right) - \ln\left(1-\frac{r}{c_2}\right)\right)\right)$$

or in its expanded form,

$$= \frac{1}{\sqrt{4-\alpha^2}}\left(\mathrm{Li}_2\left(\frac{r}{1-\frac{\alpha i}{2}+\frac{1}{2}\sqrt{4-\alpha^2}}\right) - \mathrm{Li}_2\left(\frac{r}{1-\frac{\alpha i}{2}-\frac{1}{2}\sqrt{4-\alpha^2}}\right)\right.$$
$$\left. + \ln(r)\left(\ln\left(1-\frac{r}{1-\frac{\alpha i}{2}+\frac{1}{2}\sqrt{4-\alpha^2}}\right) - \ln\left(1-\frac{r}{1-\frac{\alpha i}{2}-\frac{1}{2}\sqrt{4-\alpha^2}}\right)\right)\right)$$

Evaluating $\int_{C_3} f(z)dz$,

$$\int_{C_3} f(z)\, dz = \lim_{R\to\infty} \int_0^{\arccos\frac{1}{R}} \frac{-(-t + \ln(R)i)Re^{ti}}{-R^2e^{2ti} + (2-\alpha i)Re^{ti} + \alpha i}\, dt = 0$$

Since the denominator of the integrand has a higher degree of R, Using partial fraction decomposition in a similar manner to the integral C_2,

$$= \int_0^{\arccos \frac{1}{r}} \frac{1}{\sqrt{4-\alpha^2}} \left(\frac{te^{ti}}{e^{ti} - c_2} - \frac{te^{ti}}{e^{ti} - c_1} \right) dt$$

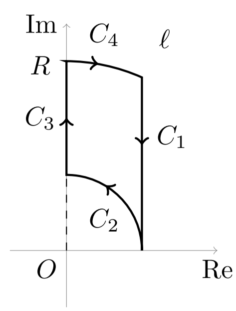
where $\ell \to \infty$.

$$\int_{C_3} f(z)dz = \int_1^{\infty} \frac{-\frac{\pi}{2} + \ln(r)i}{t^2 + (\alpha + 2i)t + \alpha i}\, dt$$

$$\begin{aligned}
\int_{-\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{C}_2} f(z)\,dz = 0 &\implies \int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz \\
&\implies \int_0^{\infty} \frac{\ln|1+it| + i\arctan(t)}{t^2 + 2\alpha t + 1}\, i\,dt = \int_1^{\infty} \frac{\ln(t)}{\alpha i + (2-\alpha i)t - t^2}\, dt \\
&\implies \int_0^{\infty} \frac{\arctan(t)}{t^2 + 2\alpha t + 1}\, dt = -\left(\int_1^{\infty} \frac{\ln(t)}{\alpha i + (2-\alpha i)t - t^2}\, dt \right) \\
&\qquad\qquad = -\int_1^{\infty} \frac{\ln(t)\left(2t - t^2\right)}{\left(2t - t^2\right)^2 + \alpha^2 (t-1)^2} dt
\end{aligned}$$

This is a significant result because we have redefined the original integral as the real component of the integral along G2. Although its form proved too difficult for us to simplify, it suggests the capability to use a similar approach for other integrals with arctangent in a rational expression.

2.3 Second Contour Attempted



where $R \to \infty$,
$C_1 = 1 + ti$ for $t \in [0, \sqrt{r^2 - 1}]$,
$C_2 = e^{ti}$ for $t \in [0, \frac{\pi}{2}]$,
$C_3 = ti$ for $t \in [1, r]$,
$C_4 = Re^{ti}$ for $t \in [\arccos \frac{1}{r}, \frac{\pi}{2}]$,

The integral along C_1 resolves in the same manner as shown in 2.2. The integral along C_4 is also equal to 0 for the same reason the integral along C_3 in the previous section was, which is that the integrand approaches 0 as $R \to \infty$.

Simplifying the integral along C_2,

$$\int_{C_3} f(z)\,dz = \int_0^{\arccos \frac{1}{r}} \frac{(ti)e^{ti}}{-e^{2ti} + (2-\alpha i)e^{ti} + \alpha i} \cdot (ie^{ti})\, dt$$

$$= \int_0^{\arccos \frac{1}{r}} \frac{-te^{2ti}}{-e^{2ti} + (2-\alpha i)e^{ti} + \alpha i}\, dt$$

Solving this integral either requires dilogarithms, which resulted in the indefinite integral,

$$= \frac{1}{\sqrt{4-\alpha^2}} \left(\mathrm{Li}_2 \left(\frac{r}{1 - \frac{\alpha i}{2} + \frac{1}{2}\sqrt{4-\alpha^2}} \right) - \mathrm{Li}_2 \left(\frac{r}{1 - \frac{\alpha i}{2} - \frac{1}{2}\sqrt{4-\alpha^2}} \right) \right.$$
$$\left. + \ln(r) \left(\ln \left(1 - \frac{r}{1 - \frac{\alpha i}{2} + \frac{1}{2}\sqrt{4-\alpha^2}} \right) - \ln \left(1 - \frac{r}{1 - \frac{\alpha i}{2} - \frac{1}{2}\sqrt{4-\alpha^2}} \right) \right) \right)$$

Which we weren't able to simplify after plugging in the bounds, or using Euler's formula to expand the expressions within the integrand, which we were not able to solve either.

When trying to solve for the integral along C_3,

$$\int_{C_2} f(z)dz = \int_1^\infty \frac{\ln(ti)}{-(ti)^2 + (2-\alpha i)(ti) + \alpha i} \cdot (i)\, dt$$
$$= \int_1^\infty \frac{-\frac{\pi}{2} + \ln(t) i}{t^2 + (2i+\alpha)t + \alpha i}\, dt$$

We initially made the mistake of assuming that taking the real part of the integral would yield $\int_1^\infty \frac{-\frac{\pi}{2}}{t^2+(2i+\alpha)t+\alpha i}\, dt$, which we could easily solve using partial fraction decomposition. However, this does not work because the denominator has real and complex parts, so we need to multiply the denominator and numerator by the conjugate of the denominator to make the denominator real, in order to be able to take the real part of the expression. Doing this step and taking the real part results in an logarithm remaining in the numerator, so were not able to simplify the expression.

3 Indefinite Integral

3.0 Definitions

Definition 1. The dilogarithm, denoted as $\mathrm{Li}_2(u)$ for a variable $u \in \mathbb{C}$, can be defined as

$$\mathrm{Li}_2(u) = -\int \frac{\ln(1-u)}{u}\,du$$

Theorem 1. For a given variable $x \in \mathbb{C}$, and a constant $c \in \mathbb{C}$,

$$\int \frac{\ln x}{x-c}\,dx = \mathrm{Li}_2\left(\frac{x}{c}\right) + \ln(x)\ln\left(1-\frac{x}{c}\right) + C$$

Proof:

First, rewrite the expression in order to use integration by parts.

$$\int \frac{\ln x}{x-c}\,dx = -\frac{1}{c}\int \frac{\ln x}{1-\frac{x}{c}}\,dx$$

Let $u = \ln x$ and $dv = -\frac{1}{c}\cdot\frac{1}{1-\frac{x}{c}}\,dx$. Then, $du = \frac{1}{x}\,dx$ and $v = \ln\left(1-\frac{x}{c}\right)$. Therefore,

$$\int \frac{\ln x}{x-c}\,dx = \ln(x)\ln(1-\frac{x}{c}) - \int \frac{\ln(1-\frac{x}{c})}{x}\,dx$$

Let $t = \frac{x}{c}$, so $dt = \frac{1}{c}\,dx$,

$$\int \frac{\ln(1-\frac{x}{c})}{x}\,dx = \int \frac{\ln(1-t)}{t\cdot c}\cdot c\,dt$$

$$= \int \frac{\ln(1-t)}{t}\,dt = \mathrm{Li}_2(t) + C$$

$$= \mathrm{Li}_2\left(\frac{x}{c}\right) + C$$

Theorem 2. For a given variable $x \in \mathbb{R}$,

$$\arctan x = \frac{1}{2i}\ln\left(\frac{i-x}{i+x}\right)$$

Proof:

Let $\arctan x = z \in \mathbb{Z}$. Therefore,

$$x = \tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\cdot\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = \frac{1}{i}\cdot\frac{e^{2iz}-1}{e^{2iz}+1}$$

Solving for z,

$$ix = \frac{e^{2iz}-1}{e^{2iz}+1} \to e^{2iz} = \frac{1+ix}{1-ix} = \frac{i-x}{i+x}$$

$$2iz = \ln\left(\frac{i-x}{i+x}\right) \to z = \frac{1}{2i}\ln\left(\frac{i-x}{i+x}\right)$$

.

3.1 Solving the Intgeral

The original integral can be expressed differently using Theorem 2:

$$\int \frac{\arctan x}{1+\alpha x+x^2}\,dx = \int \frac{\frac{1}{2i}\ln\left(\frac{i-x}{i+x}\right)}{1+\alpha x+x^2}\,dx$$

$$= \frac{1}{2i}\left(\int \frac{\ln(i-x)}{1+\alpha x+x^2}\,dx - \int \frac{\ln(i+x)}{1+\alpha x+x^2}\,dx\right)$$

Letting $x_1 = i - x$ and $x_2 = i + x \to x = i - x_1 = x_2 - i$, the expression simplifies to:

$$= \frac{1}{2i}\left(\int \frac{\ln(x_1)}{x_1^2+(-\alpha-2i)x_1+\alpha i}\,dx - \int \frac{\ln(x_2)}{x_2^2+(\alpha-2i)x_2-\alpha i}\,dx\right)$$

Splitting up the integrals with partial fraction decomposition,

$$\begin{aligned}
= \frac{1}{2i}\Bigg(&\frac{1}{\sqrt{\alpha^2-4}}\int \frac{\ln(x_1)}{x_1-\left(i+\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}\right)} - \frac{\ln(x_1)}{x_1-\left(i+\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}\right)}\,dx\\
&-\frac{1}{\sqrt{\alpha^2-4}}\int \frac{\ln(x_2)}{x_2-\left(i-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}\right)} - \frac{\ln(x_2)}{x_2-\left(i-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}\right)}\,dx\Bigg)
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{2\sqrt{4-\alpha^2}}\Bigg(&\int \frac{\ln(x_1)}{x_1-\left(i+\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}\right)} - \frac{\ln(x_1)}{x_1-\left(i+\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}\right)}\,dx\\
&-\int \frac{\ln(x_2)}{x_2-\left(i-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}\right)} - \frac{\ln(x_2)}{x_2-\left(i-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}\right)}\,dx\Bigg)
\end{aligned}$$

Using Theorem 1 to simplify the integrals,

$$\begin{aligned}
= \frac{1}{2\sqrt{4-\alpha^2}}\Bigg(&\mathrm{Li}_2\left(\frac{x_1}{i+\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}}\right) - \mathrm{Li}_2\left(\frac{x_1}{i+\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}}\right)\\
&+\ln(x_1)\left(\ln\left(1-\frac{x_1}{i+\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}}\right) - \ln\left(1-\frac{x_1}{i+\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}}\right)\right)\\
&\mathrm{Li}_2\left(\frac{x_2}{i-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}}\right) - \mathrm{Li}_2\left(\frac{x_2}{i-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}}\right)\\
&+\ln(x_2)\left(\ln\left(1-\frac{x_2}{i-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4}}{2}}\right) - \ln\left(1-\frac{x_2}{i-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4}}{2}}\right)\right)\Bigg) + C
\end{aligned}$$

Plugging x_1 and x_2 back in and simplifying further,

$$\begin{aligned}
= \frac{1}{2\sqrt{4-\alpha^2}}\Bigg(&\mathrm{Li}_2\left(\frac{2i-2x}{2i+\alpha+\sqrt{\alpha^2-4}}\right) - \mathrm{Li}_2\left(\frac{2i-2x}{2i+\alpha-\sqrt{\alpha^2-4}}\right)\\
&+\ln(i-x)\left(\ln\left(1-\frac{2i-2x}{2i+\alpha+\sqrt{\alpha^2-4}}\right) - \ln\left(1-\frac{2i-2x}{2i+\alpha-\sqrt{\alpha^2-4}}\right)\right)\\
&+\mathrm{Li}_2\left(\frac{2i+2x}{2i-\alpha-\sqrt{\alpha^2-4}}\right) - \mathrm{Li}_2\left(\frac{2i+2x}{2i-\alpha+\sqrt{\alpha^2-4}}\right)\\
&+\ln(i+x)\left(\ln\left(1-\frac{2i+2x}{2i-\alpha-\sqrt{\alpha^2-4}}\right) - \ln\left(1-\frac{2i+2x}{2i-\alpha+\sqrt{\alpha^2-4}}\right)\right)\Bigg) + C
\end{aligned}$$

We were not able to simplify the expression after plugging in the bounds, but this may be a possible direction for future exploration.

4 Discussion

4.0 Future Work

While C_1 and C_3 from the second contour were promising, we need two curves to close the contour which we can simplify, and they must be in the positive imaginary plane.
In the future we will consider a new geometry for the contour, and possibly allow poles within the region, exploring curves with negative imaginary components. Another idea is a contour that depends on the variable α to simplify based on how the integral shifts with different α

Furthering our understanding of properties and limits of dilogarithms could prove useful in continuing to look at the indefinite integral in an attempt to continue manipulating or evaluating it.

With investigating behaviors of the dilogarithm and possible alternative substitutions or contours to test, we are yet to find a solution. Contour integration is promising due to simplifying the integral itself and separating it into a simple RHS and our choice of a LHS so that we can manipulate it to best substitute for the simplest answer.

4.1 Conclusion

While these specific contours did not lead to nicely simplified solutions, aspects of each contour were valuable in looking at end behaviors especially those involving curves that the integral approaches 0, or looking at how the problem reacts when the contour you are integrating along lies directly on either the real axis or the imaginary axis.

Also, the investigation of dilogarithm in the future could prove useful as polylogarithms often appear in complex analysis.

Fundamentally, our approach allows for a general way to rewrite any integrals containing arctangent by integrating along the contour $z = 1 + ti$. This substitution and parametrization could allow us to apply this technique to various problems. Furthermore, familiarizing ourselves with concepts of complex analysis and specifically the Cauchy Residue Thorem with contour integration is valuable to apply to similar problems that involve the complex plane.

Acknowledgments

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References

[1] Eric W. Weisstein. Dilogarithm. 2024.